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The slow motion of a sphere in a rotating, viscous fluid

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The uniform, slow motion of a sphere in a viscous fluid is examined in the case where the undisturbed fluid rotates with constant angular velocity Ω and the axis of rotation is taken to coincide with the line of motion. The various modifications of the classical problem for small Reynolds numbers are discussed. The main analytical result is a correction to Stokes's drag formula, valid for small values of the Reynolds number and Taylor number and tending to the classical Oseen correction as the last parameter tends to zero. The rotation of a free sphere relative to the fluid at infinity is also deduced.

1. Introduction

In this paper we give a simple extension to rotating flows of the classical low-Reynolds-number theory for infinite three-dimensional regions. The perturbation method which is used is due to Kaplun & Lagerstrom (1957). A problem in magnetohydrodynamics, similar in many respects to that discussed here, has been treated by Chester (1957) and, using the methods of the present paper, by Chang (1960). The suggestion that this similarity should extend to the perturbation procedure was made by Maxworthy (1962).

The following specific problem is examined. Consider a fluid of constant density ρ and kinematic viscosity ν, which is in solid-body rotation with angular velocity Ω. A sphere of radius a moves with speed U along the axis of rotation, and is free to rotate about the same axis. An approximate description of the flow pattern is sought, which is valid in the asymptotic sense for small values of the Reynolds number $Re = Ua/\nu$ and Taylor number $Ta = \Omega a^2/\nu$. Such an approximation can be obtained (cf. §6) by expansion with respect to Re alone, with a new parameter α,

$$\alpha = 2(Ta/Re^2) = 2(\Omega\nu/U^2),$$

held fixed. The principal results of the present investigation may be summarized as follows. If D is the drag experienced by the sphere, and if ω is the angular velocity of the sphere relative to the fluid at infinity, then the expansions with respect to Re are

$$D/6\pi\rho\nu Ua = 1+\lambda(\alpha)\,Re+o(Re), \tag{1a}$$

$$\omega/\Omega = \chi(\alpha)\,Re+o(Re), \tag{1b}$$

where the functions $\lambda(\alpha)$ and $\chi(\alpha)$ are given by the definite integrals

$$\lambda(\alpha) = \frac{3}{4}\int_0^1 \{(t^2+4i\alpha t)^{\frac{1}{2}}+(t^2-4i\alpha t)^{\frac{1}{2}}\}(3t^2-1)\,dt, \tag{2a}$$

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$$\chi(\alpha) = \frac{3i}{8\alpha}\int_0^1 \{(t^2+4i\alpha t)^{\frac{1}{2}} - (t^2-4i\alpha t)^{\frac{1}{2}}\}(3t^3-t)\,dt. \tag{2b}$$

(Numerical values and series expansions are given below in §4.)

The derivation of (1), (2) is described in §§2–4.† In §2 the main elements of the perturbation method are reviewed, in order to point out the modifications that are required to account for rotation. It is found that the effect of rotation is twofold. First, there is near the sphere an added acceleration caused by the Coriolis force. This acceleration is $O(Ta)$ and therefore is here of higher order than the non-linear effect (the added acceleration due to the non-linear terms) which is $O(Re)$. Secondly, in the outer flow field, a distance $O(Re^{-1})$ from the sphere, the Coriolis term is of the same order of magnitude as the viscous and convective terms and the *perturbation* caused by the sphere is consequently altered. This last effect changes the values of the velocity perturbation observed near the sphere, and therefore appears in the computation of the second-order velocity there as a change in the boundary conditions at 'infinity'. To compute these new conditions (actually matching conditions) a Fourier representation of the outer solution will be used and is derived in §3. Finally, in §4, the computation of λ and χ is completed by showing how these numbers can be extracted from our partial knowledge of the flow field.

In §5 we consider briefly the effect of rotation upon the wake structure far from the sphere. It is shown that the perturbation there consists of symmetric diffusive wakes extending fore and aft of the sphere. The transverse dimension of these wakes grows as the one-third power of the distance, in contrast with the one-half power dependence of the viscous wave of the Oseen theory.

2. Asymptotic expansions in *Re*

The equations appropriate to the physical problem described in §1 are (in dimensionless notation with reference velocity U and length a)

$$Re\,\mathbf{q}\,.\,\nabla\mathbf{q} + \nabla p + 2Ta\,\mathbf{i}\times\mathbf{q} - \nabla^2\mathbf{q} = 0, \tag{3a}$$

$$\nabla\,.\,\mathbf{q} = 0. \tag{3b}$$

Since the sphere is free to rotate about the axis of symmetry, the boundary conditions are

$$\mathbf{q} = (a\omega/U)\,\mathbf{i}\times\mathbf{r} \quad \text{when} \quad r = (x^2+y^2+z^2)^{\frac{1}{2}} = 1; \tag{3c}$$

$$\mathbf{q} = \mathbf{i}, \quad p = 0 \quad \text{when} \quad r = \infty. \tag{3d}$$

In the above, the co-ordinate system moves with the sphere, and rotates with the undisturbed fluid so that the term in (3a) containing Ta has its origin in the Coriolis force experienced by a fluid element. The quantity p is defined by

$$p = (a/\rho\nu U)\,[p^* - \tfrac{1}{2}\rho\Omega^2 a^2(y^2+z^2)],$$

where p^* is the pressure. In (3c), ω is the dimensional angular velocity of the sphere relative to the rotating co-ordinate system. This parameter is not known in

† An earlier analysis by the author was devoted exclusively to the drag correction. The relative rotation of a free sphere was subsequently deduced by Saffman (1963), to whom the author is indebted for permission to include his results in the present paper, and also for comments on a previous draft.

advance but will be determined (as a function of Re and Ta) by the requirement that the torque on the sphere be zero.

In the present paper we seek to find an approximation to $\mathbf{q}, p$ in $r \geqslant 1$, valid for small Re, with the additional stipulation that $Ta = Ta(Re) = O(Re^2) = \frac{1}{2}\alpha Re^2$. This can be accomplished with the help of known expansion procedures, since the rotating ($\alpha > 0$) and non-rotating ($\alpha = 0$) cases of our problem are very similar in form. Following Kaplun & Lagerstrom (1957), we shall consider inner and outer expansions in Re, having the respective forms

$$\mathbf{q}(\mathbf{r}; Re) = \mathbf{q}_0(\mathbf{r}) + Re\,\mathbf{q}_1(\mathbf{r}) + o(Re) \quad (1 \leqslant r < \infty), \tag{4a}$$

$$\mathbf{q}(\mathbf{r}; Re) = \mathbf{i} + Re\,\mathbf{q}'(\tilde{\mathbf{r}}) + o(Re) \qquad (\tilde{r} > 0), \tag{4b}$$

where in (4b) the outer variables $\tilde{x} = xRe, \tilde{y} = yRe, \tilde{z} = zRe$ are used. Expansions for the pressure are similar and will be omitted. If $\alpha > 0$ inner and outer expansions of the axial vorticity ζ,

$$\zeta = \mathbf{i}.\nabla \times \mathbf{q},$$

of the respective forms

$$\zeta(\mathbf{r}; Re) = Re^2\zeta_2(\mathbf{r}) + o(Re^2) \quad (1 \leqslant r < \infty), \tag{5a}$$

$$\zeta(\mathbf{r}; Re) = Re^2\zeta_2'(\mathbf{r}) + o(Re^2) \quad (\tilde{r} > 0), \tag{5b}$$

must also be considered, as will be apparent below.

The construction of (4) and (5) will now be reviewed and proceeds as follows. The leading terms $\mathbf{q}_0$, p_0 can be shown to be solutions of Stokes's problem,

$$\nabla p_0 - \nabla^2\mathbf{q}_0 = 0, \quad \nabla.\mathbf{q}_0 = 0, \tag{6a}$$

$$p_0 = 0, \quad \mathbf{q}_0 = \mathbf{i} \quad \text{when} \quad r = \infty, \tag{6b}$$

$$\mathbf{q}_0 = 0 \quad \text{when} \quad r = 1. \tag{6c}$$

In particular the matching conditions (6b) are unchanged by rotation to this order. The inner boundary condition (6c) states that the sphere does not rotate differentially to this order. To see that this must be so, suppose that a term of order unity proportional to $\mathbf{i} \times \mathbf{r}$ is added to $\mathbf{q}_0$ as defined above, so as to satisfy a boundary condition for differential rotation. The resulting term of order unity in the inner expansion of ζ must then be matched with a corresponding outer term. This clearly is not possible, since far from the sphere the perturbation is small and the fluid moves axially. In order to satisfy the matching condition on ζ, it is therefore necessary to add to $\mathbf{q}_0$ a second term which satisfies a null condition on the sphere and cancels the solid-body rotation at infinity. The sum of the two added terms thus provides a solution of the Stokes equations which represents the flow caused by a sphere spinning in a fluid at rest, and this requires a torque of order unity. Our previous condition, that the torque be zero to all orders in Re, eliminates this possibility and we conclude that $\mathbf{q}_0$ must satisfy (6c). This is an obvious result at this stage, but essentially the same argument may be used for higher-order terms.

The first-order outer terms $\mathbf{q}'$, p' satisfy the Oseen equations, containing now the Coriolis term

$$\frac{\partial \mathbf{q}'}{\partial \tilde{x}} + \tilde{\nabla} p' + \alpha \mathbf{i} \times \mathbf{q}' - \tilde{\nabla}^2 \mathbf{q}' = 0, \tag{7a}$$

$$\tilde{\nabla}.\mathbf{q}' = 0. \tag{7b}$$

The outer boundary conditions are

$$\mathbf{q}' = 0, \quad p' = 0 \quad \text{when} \quad \tilde{r} = \infty, \tag{7c}$$

and there is in addition a matching condition at $\tilde{r} = 0$. There are several equivalent ways of stating this last matching condition. In this paper we shall require that $\mathbf{q}'$ have at $\tilde{r} = 0$ the singularity of a 'fundamental solution' of (7) corresponding to a force equal to the Stokes drag of the sphere. Such a solution may be obtained formally by solving (7) with the right-hand side of (7a) replaced by $-6\pi\delta(\tilde{\mathbf{r}})\,\mathbf{i}$. That this is a sufficient condition on $\mathbf{q}'$ can be seen by the following argument. The precise matching condition states that a certain part of $\mathbf{q}_0$, which dominates this term in some intermediate region where the matching condition is applied (the overlap domain), is cancelled there by a part of $\mathbf{i}+Re\,\mathbf{q}'$. The common part of these two terms may be shown to be equal to a fundamental solution of the *Stokes* equations, corresponding as before to the Stokes drag. If we denote this common part by $\mathbf{A}$, the matching condition then states that $\mathbf{i}+Re\,\mathbf{q}'-\mathbf{A}$ is bounded when $\tilde{\mathbf{r}}$ is small, i.e. in a region where the Stokes equations approximate the Oseen equations. Thus the singularity is the same in either case. The physical meaning of this is that in the overlap domain the sphere has the same effect as a point disturbance.

The inner terms of order Re satisfy

$$\nabla p_1 - \nabla^2 \mathbf{q}_1 = -\mathbf{q}_0 . \nabla \mathbf{q}_0, \quad \nabla . \mathbf{q}_1 = 0. \tag{8a}$$

The matching condition is obtained by writing

$$\mathbf{i} + Re\,\mathbf{q}' - \mathbf{A} = Re\{\mathbf{B}(\alpha) + o(1)\} \quad \text{as} \quad \tilde{r} \to 0,$$

$$\mathbf{A} = \mathbf{i} - \frac{3}{2}\left(\frac{\mathbf{i}}{r} - \nabla \frac{x}{2r}\right),$$

where $\mathbf{B}(\alpha)$ is also dependent on the direction of $\tilde{r}$ (cf. § 4). Then it is required that

$$\lim_{Re \to 0} \quad \mathbf{q}_1 = \mathbf{B}(\alpha), \tag{8b}$$

where the co-ordinates lie in some overlap domain. The condition that the torque on the sphere vanishes to order Re inclusive implies, in the same way as before, that the term $\mathbf{q}_1$ satisfies the null condition

$$\mathbf{q}_1 = 0 \quad \text{when} \quad r = 1. \tag{8c}$$

The first non-trivial term in the inner expansion of ζ is of order Re^2, and there are two contributions. First, the Coriolis force associated with $\mathbf{q}_0$ is of this order. Secondly, the outer term $\mathbf{q}'$ introduces, through the matching condition, a non-zero term of this order. Using (5) in the equation for the vorticity

$$\nabla^2 \zeta_2 = -\alpha \mathbf{i} . \partial \mathbf{q}_0 / \partial x, \tag{9a}$$

$$\lim_{Re \to 0} \quad \zeta_2(\mathbf{r}) = C(\alpha), \tag{9b}$$

say, in some overlap domain, where

$$\zeta_2'(\tilde{\mathbf{r}}) = C(\alpha) + o(1) \quad \text{as} \quad \tilde{r} \to 0. \tag{10}$$

From the above we may draw an important conclusion. To order Re in velocity and to order Re^2 in ζ, i.e. to first order in each case, the effects of rotation lie entirely in the appearance of the Coriolis term as a forcing term in (9*a*), the altered matching condition (8*b*), and the new matching condition (9*b*). This means that, for the purpose of obtaining the inner flow field to these orders, and subsequently to carry out the computation of λ and χ, only an expansion of $\mathbf{q}'$ for small $\tilde{r}$ is actually used to extract the quantities $\mathbf{B}(\alpha)$ and $C(\alpha)$. This will allow us to simplify the analysis considerably, since this expansion can be found, for example, directly from a Fourier representation of $\mathbf{q}'$. We now turn to this problem.

3. The fundamental solution

We may solve (7), now with $-6\pi\delta(\tilde{\mathbf{r}})\,\mathbf{i}$ on the right-hand side of (7*a*), by introducing the three-dimensional Fourier transforms $\mathbf{\Gamma}(\mathbf{k})$, $\Pi(\mathbf{k})$ defined by

$$\mathbf{q}' = \frac{1}{8\pi^3}\int e^{i\mathbf{k}\cdot\tilde{\mathbf{r}}}\,\mathbf{\Gamma}(\mathbf{k})\,d\mathbf{k}, \tag{11a}$$

$$p' = \frac{1}{8\pi^3}\int e^{i\mathbf{k}\cdot\tilde{\mathbf{r}}}\,\Pi(\mathbf{k})\,d\mathbf{k}. \tag{11b}$$

Substitution of (11) in (7) leads formally to the system

$$ik_1\mathbf{\Gamma} + i\mathbf{k}\Pi + \alpha\mathbf{i}\times\mathbf{\Gamma} + k^2\mathbf{\Gamma} = -6\pi\mathbf{i}, \tag{12a}$$

$$\mathbf{k}\cdot\mathbf{\Gamma} = 0, \quad \mathbf{k} = k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k}', \tag{12b}$$

for $\mathbf{\Gamma}$ and Π. The cross-product of $\mathbf{k}$ and the first of these equations, taken twice, yields two equations which may be solved for $\mathbf{\Gamma}$. Thus

$$\mathbf{\Gamma}(\mathbf{k}) = -6\pi\left[\frac{(k^2\mathbf{i}-k_1\mathbf{k})\,(k^2+ik_1) - \alpha k_1(\mathbf{k}\times\mathbf{i})}{(k^2+ik_1)^2\,k^2 + \alpha^2k_1^2}\right], \tag{13a}$$

$$\Pi(\mathbf{k}) = i/k^2[6\pi k_1 + \alpha(\mathbf{k}\times\mathbf{i})\cdot\mathbf{\Gamma}]. \tag{13b}$$

In order to prove that (11), (13) define the outer terms we must show that (i) $\mathbf{q}'$ plus 6π times the fundamental solution of the Stokes equations is bounded in a neighbourhood of the origin; (ii) the boundary conditions (7*c*) are satisfied; and (iii) the terms in (7*a*) and (7*b*) exist and the equations are satisfied when $\tilde{r} > 0$. To show (iii) it suffices to consider the partial inversion of $\mathbf{\Gamma}$ or Π with respect to k_1. We use the theory of residues and evaluate the integrand at roots k_1 of

$$(k^2+ik_1)^2\,k^2 + \alpha^2k_1^2 = 0, \tag{14}$$

which lie in a suitable half-plane. It is easily seen that a root of (14) has non-zero imaginary part, uniformly on $0 < \delta \leqslant k_2^2 + k_3^2 \leqslant \infty_1$. The residues corresponding to those roots which have positive imaginary part therefore vanish exponentially as $k_2^2 + k_3^2 \to \infty$ uniformly for $x \geqslant \epsilon > 0$, where ϵ is arbitrarily small. The integral with respect to k_2 and k_3 converges, along with all derivatives, absolutely, and (iii) may therefore be established from this representation.

In order to prove (i) and (ii), we first define $\mathbf{\Gamma}_S$, Π_S, the Fourier transform of the Stokes solution $(\mathbf{A}-\mathbf{i})/Re$ (cf. § 2), by

$$i\mathbf{k}\Pi_S + k^2\mathbf{\Gamma}_S = -6\pi\mathbf{i}, \tag{15a}$$

$$\mathbf{k}\cdot\mathbf{\Gamma}_S = 0. \tag{15b}$$

Thus
$$\mathbf{\Gamma}-\mathbf{\Gamma}_S = 6\pi\left[\frac{(k^2\mathbf{i}-k_1\mathbf{k})\,(ik_1k^4-k_1^2k^2+\alpha^2k_1^2)+\alpha k_1k^4(\mathbf{k}\times\mathbf{i})}{(k^2+ik_1)^2k^6+\alpha^2k_1^2k^4}\right], \tag{16a}$$
$$\Pi-\Pi_S = \alpha(\mathbf{k}\times\mathbf{i}).\mathbf{\Gamma}/k^2. \tag{16b}$$

It is seen from (16) that there are constants M, N such that, for $\alpha > 0$, $0 < k \leqslant \infty$, we have the estimates
$$|\mathbf{\Gamma}-\mathbf{\Gamma}_S| \leqslant M/k^2(1+k), \tag{17a}$$
$$|\Pi-\Pi_S| \leqslant N/k^2(1+k). \tag{17b}$$

That $\mathbf{q}'$ and p' vanish at infinity follows from (17), using the Riemann–Lebesgue lemma, and from the fact that the Stokes solution defined above vanishes there. Finally, it can be shown from a direct calculation (which we carry out in §4) that the integral of $(\mathbf{\Gamma}-\mathbf{\Gamma}_S)\,e^{i\mathbf{k}.\tilde{\mathbf{r}}}$ with respect to $\mathbf{k}$ is bounded at the origin.

4. Computation of λ and χ

In the present paragraph we shall show how the numbers $\lambda(\alpha)$ and $\chi(\alpha)$ can be found from the Fourier representation of $\mathbf{q}'$. In doing so only a part of the inner expansion of contributing order need be considered, and we begin with several observations concerning this point.

It is known from the results of Kaplun & Lagerstrom (1957), Chester (1962) and others that the Navier–Stokes and Oseen expansions of, for example, the drag agree to order *Re* inclusive for certain classes of solids. In particular, for a sphere, that part of $\mathbf{q}_1$ which is associated with the forcing term $-\mathbf{q}_0.\nabla\mathbf{q}_0$ does not make a contribution to the drag, as can be seen from a symmetry argument. Let us call a flow field $\mathbf{q}$ odd if the axial component is odd in x. Then the symmetry argument states that in the axially symmetric problem under consideration an odd term in $\mathbf{q}_1$ cannot alter the drag of a solid symmetric about $x = 0$. This eliminates not only the particular solution of (8a), but also odd terms which are induced by the matching condition (8b). Similarly, in the inner term ζ_2, a part which is odd in x cannot cause a solid symmetric about the plane $x = 0$ to rotate differentially. This eliminates from our discussion the part generated by the forcing term in (9a), as well as odd terms induced by the matching condition (9b). The computation of λ and χ therefore depends solely upon the homogeneous solutions of (8a) and (9a) which match with the even part of $\mathbf{B}$ and C, and satisfy the inner boundary condition.

Now, by definition,
$$(1/8\pi^3)\int(\mathbf{\Gamma}-\mathbf{\Gamma}_S)\,e^{i\mathbf{k}.\tilde{\mathbf{r}}}d\mathbf{k} = \mathbf{B}(\alpha)+o(1),$$
$$(i/8\pi^3)\int\mathbf{i}.(\mathbf{k}\times\mathbf{\Gamma})\,e^{i\mathbf{k}.\tilde{\mathbf{r}}}d\mathbf{k} = C(\alpha)+o(1),$$
as $\tilde{r}\to 0$. To evaluate the terms on the left we shall divide up the region of integration into two parts, $0 \leqslant k \leqslant \tilde{r}^{-\sigma}$, and $k > \tilde{r}^{-\sigma}$, where $0 < \sigma < 1$.† Then, using (17), we have
$$\frac{1}{8\pi^3}\int(\mathbf{\Gamma}-\mathbf{\Gamma}_S)\,e^{i\mathbf{k}.\tilde{\mathbf{r}}}d\mathbf{k} = \frac{1}{8\pi^3}\int_{k\leqslant\tilde{r}^{-\sigma}}(\mathbf{\Gamma}-\mathbf{\Gamma}_S)\,d\mathbf{k}+\frac{i}{8\pi^3}\int_{k>\tilde{r}^{-\sigma}}\frac{(k^2\mathbf{i}-k_1\mathbf{k})\,k_1}{k^6}$$
$$\times\, e^{i\mathbf{k}.\tilde{\mathbf{r}}}d\mathbf{k}+O(\tilde{r}^{1-\sigma})+O(\tilde{r}^{\sigma}) \tag{18}$$

† This step was suggested to the author by a referee.

as $\tilde{r} \to 0$. The second term on the right-hand side of (18) is an odd term, while the first is parallel to $\mathbf{i}$. In the limit there is obtained

$$\mathbf{B}(\alpha) = \mathbf{i}\frac{3}{4\pi^2}\int \frac{(k^2-k_1^2)(ik_1k^4-k_1^2k^2+\alpha^2k_1^2)}{(k^2+ik_1)k^6+\alpha^2k_1^2k^4}\,d\mathbf{k}+\ldots, \tag{19a}$$

where the dots indicate an odd remainder and integration is now over all $\mathbf{k}$. The computation of $C(\alpha)$ is similar and there results

$$C(\alpha) = \frac{3i}{4\pi^2}\alpha\int \frac{k_1(k^2-k_1^2)}{(k^2+ik_1)^2k^2+\alpha^2k_1^2}\,d\mathbf{k}+\ldots, \tag{19b}$$

where again the omitted term is odd.

The passage from (19) to (1) is now straightforward. That part of $\mathbf{q}_1$ which matches the even part of $\mathbf{B}$ is clearly proportional to $\mathbf{q}_0$, and there is a proportional increment in the drag. Thus λ is equal to the coefficient of $\mathbf{i}$ displayed on the right of (19a). That part of ζ_2 which matches the even part of C will contribute a torque unless it reduces to a constant, representing a solid-body rotation of the inner flow field with dimensionless angular velocity $\frac{1}{2}Re^2\alpha\chi(\alpha)$, according to (1$b$). It follows that the term displayed on the right of (19b) is equal to $\alpha\chi(\alpha)$. The more useful expressions (2) given above may then be obtained from the integrals over $\mathbf{k}$ in (19) by the introduction of spherical co-ordinates and contour integration with respect to k.

For $4\alpha > 1$ we have the expansions

$$\lambda(\alpha) = \frac{2\sqrt{(2\alpha)}}{7}\left(1+\frac{7}{40}\frac{1}{\alpha}+\frac{15}{1408}\frac{1}{\alpha^2}-\frac{49}{39\,936}\frac{1}{\alpha^3}-\ldots\right), \tag{20a}$$

$$\chi(\alpha) = -\frac{\sqrt{2}}{5\sqrt{\alpha}}\left(1-\frac{75}{616}\frac{1}{\alpha}+\frac{35}{4992}\frac{1}{\alpha^2}-\frac{45}{28\,160}\frac{1}{\alpha^3}+\ldots\right). \tag{20b}$$

For values of α between 0 and 1, tables 1 and 2 may be used. It is interesting to note that, for sufficiently small values of α, the effect of rotation is actually to

α	$\lambda(\alpha)$	α	$\lambda(\alpha)$
0	0·375	0·40	0·379
0·05	0·367	0·45	0·386
0·10	0·359	0·50	0·395
0·15	0·356	0·60	0·411
0·20	0·356	0·70	0·429
0·25	0·360	0·80	0·445
0·30	0·365	1·0	0·479
0·35	0·371	—	—

TABLE 1

α	$-\chi(\alpha)$	α	$-\chi(\alpha)$
0	0·375	0·50	0·313
0·10	0·375	0·67	0·287
0·23	0·367	0·83	0·266
0·37	0·337	1·00	0·248

TABLE 2

decrease the drag, the minimum occurring for $\alpha = 0{\cdot}175$ approximately. The behaviour of $\lambda(\alpha)$ near $\alpha = 0$ is given by the expansion

$$\lambda(\alpha) = \tfrac{3}{8} + 3\alpha^2 \log \alpha + O(\alpha^4 \log \alpha). \tag{21}$$

The first term on the right of (21) corresponds, of course, to the classical Oseen correction.

We remark also that, according to (20*a*), when $\alpha \to \infty$, (1*a*) may be written in the form

$$\frac{D}{D_S} = 1 + \frac{2c}{21\pi} Ta^{\frac{1}{2}} + O(Ta), \tag{22}$$

where D_S is the Stokes drag and $c = D_S/\rho U\nu a$. It can be shown that (22) is valid also for any finite solid which is symmetric about the axis of rotation. This result may be compared with a formula obtained by Chang (1960) for a related problem in magnetohydrodynamics.

5. The wake structure

The structure of the fundamental solution derived in §3 is far more complicated in the rotating case than in the non-rotating case. However, if we restrict attention to the flow field at large distances, i.e. when $\tilde{r} \gg 1$, the main effect of the Coriolis force is quite easily seen. The peculiar effects of rotation on the flow field at large distances are most obvious from the approximate form of the partial-differential equation which is governing there. The full equation, satisfied in $\tilde{r} > 0$ by the velocity components and pressure, follows from (14) and is

$$\tilde{\nabla}^2 \left(\tilde{\nabla}^2 - \frac{\partial}{\partial \tilde{x}}\right) \left(\tilde{\nabla}^2 - \frac{\partial}{\partial \tilde{x}}\right) \phi + \alpha^2 \frac{\partial^2 \phi}{\partial \tilde{x}^2} = 0. \tag{23}$$

If $\alpha = 0$, it is well known that a decomposition or splitting of the solution occurs, and that part of the solution which is identified with the viscous wake satisfies the reduced equation

$$\tilde{\nabla}_\perp^2 \phi - \frac{\partial \phi}{\partial \tilde{x}} = 0, \quad \tilde{\nabla}_\perp^2 = \frac{\partial^2}{\partial \tilde{y}^2} + \frac{\partial^2}{\partial \tilde{z}^2} \tag{24}$$

when $\tilde{r} \gg 1$. If $\alpha > 0$, however, the splitting does not occur and the entire wake structure is described by solutions of the reduced equation

$$\tilde{\nabla}_\perp^6 \phi + \alpha^2 \frac{\partial^2 \phi}{\partial \tilde{x}^2} = 0 \tag{25}$$

when $\tilde{r} \gg 1$. Therefore, at large distances the flow pattern is entirely changed by the action of the Coriolis force. For example, we note that (25) admits solutions which are symmetric in $\tilde{x}$, while (24) does not; also, the width of the wake grows as $\tilde{x}^{\frac{1}{3}}$ rather than as $\tilde{x}^{\frac{1}{2}}$.

In order to determine the approximation to the fundamental solution we return now to the Fourier representation. If the velocity is expanded for large $|\tilde{x}|$, the dominant contribution to the integral with respect to k_1 occurs when the imaginary part of the root of (14) is smallest, and this implies that a neighbourhood of the origin $\mathbf{k} = 0$ is to be considered. It is next seen that in order that $\mathbf{k}$ be small and simultaneously lie on the surface defined by (14), k_1 must be of the order of the cube of the transverse components. If this approximation is made in (11*a*), there results for the axial component of velocity

$$u' = -\frac{3}{4\pi^2}\int \frac{(k^2-k_1^2)^2\,e^{i\mathbf{k}\cdot\tilde{\mathbf{r}}}}{(k^2-k_1^2)^3+\alpha^2k_1^2}\,d\mathbf{k}+o(\tilde{r}^{-1})$$

$$= -\frac{1}{2|\tilde{x}|}F(\eta)+o(\tilde{r}^{-1}),$$

where

$$F(\eta) = 3\int_0^\infty s^2e^{-s^3}J_0(\eta s)\,ds, \quad \eta = (\tilde{y}^2+\tilde{z}^2)^{\frac{1}{2}}(\alpha/|\tilde{x}|)^{\frac{1}{3}}.$$

The function $F(\eta)$ is shown in figure 1. Note that over a portion of the wake the speed exceeds the free-stream speed.

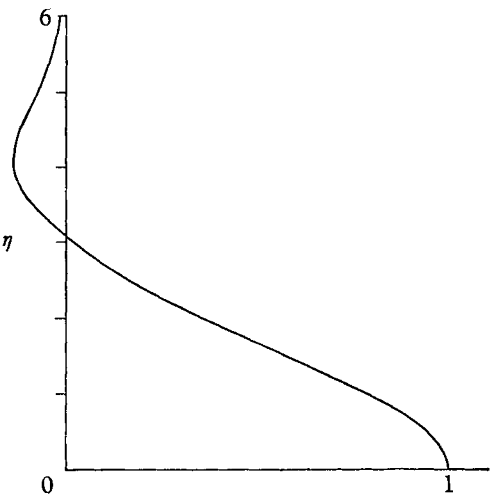

FIGURE 1. Axial velocity perturbation in the distant wake.

6. Discussion

The limit-process expansions (4), (5) have been introduced with the specific intention of finding how the rotation of the fluid affects the classical Stokes flow. Therefore, in a definite sense the present theory is 'higher order' and the effects which are calculated are in the same sense 'small' effects. (Of course, the relation between the Stokes and Oseen flows is such that arbitrarily small rotation alters the perturbation at large distances.) However, the freedom which we have in the choice of $Ta(Re)$ and the stretching of the co-ordinates would appear to offer more than one possible form for these results. We can investigate this question in physical terms by reducing all small terms to the role of forcing terms in Stokes's problem.

If we want to assess the effect of a forcing term, however small, we must consider at the same time the size of the region over which the equation is defined. The relationship between the order of magnitude of the forcing term proper, and the order of the *integrated effect* of the term, is by no means obvious in many physical problems involving infinite regions. In the present example, however, the following simple argument may be used. Let L^3 denote the volume of a closed region containing the sphere (e.g. another sphere). Consider now the convective (Oseen) term, the Coriolis term, and the viscous stress term in (3*a*). The integrated order of magnitude of these terms, taking into account the number of derivatives involved, is $L^2 Re$, $L^3 Ta$, and L, respectively. We can study the solutions in the most general way if these orders are all the same, and, if this is so, then

$$L = O(Re^{-1}), \quad Ta = O(Re^2).$$

Thus the stretching factor and the parameter α emerge from the desire that the integrated effects of all perturbations of the Stokes flow, over a region where the latter is valid, be comparable. We can simplify in various ways, by taking α small or large, without invalidating the general results, although in the case of α large it is necessary to require that

$$Ta = \tfrac{1}{2}\alpha Re^2 \ll 1 \tag{26}$$

in order to justify the division of the approximation into inner and outer expansions. (It is now apparent that, whereas it was convenient to introduce α as a fixed positive parameter, it is actually sufficient to require that (26) be satisfied.)

It is seen from these considerations that, in order to examine the effect of a dominant Coriolis term in so far as the outer problem is concerned, we may assume initially that α is large. If α is also so large that $\alpha Re \gg 1$, then this approximation can be made in the inner problem as well, so that with no further restrictions on Ta the governing equations may be taken to be

$$\nabla p + 2Ta\,\mathbf{i}\times\mathbf{q} - \nabla^2\mathbf{q}, \quad \nabla . \mathbf{q} = 0. \tag{27}$$

For these 'Stokes' equations the present paper provides approximation valid for Ta small. If Ta is not small, then the effect of the Coriolis force is of order unity over the inner flow field, and (27) must be solved with the exact boundary condition on the sphere.

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